SIMPLE EXPRESSIONS FOR THE HOLED TORUS RELATIONS

NORIYUKI HAMADA

Abstract. In the mapping class group of a k-holed torus with $0 \leq k \leq 9$, one can factorize the boundary multi-twist (or the identity when $k = 0$) as the product of twelve right-handed Dehn twists. Such factorizations were explicitly given by Korkmaz and Ozbagci for each $k \leq 9$ and an alternative one for $k = 8$ by Tanaka. In this note, we simplify their expressions for the k-holed torus relations.

Let Σ_1^k denote a torus with k boundary components and $\mathrm{Mod}(\Sigma_1^k)$ be its mapping class group where maps are assumed to be identity on the boundary. In [3], Korkmaz and Ozbagci gave a factorization of the multi-boundary twist $t_{\delta_1} \cdots t_{\delta_k}$ as the product of twelve right-handed Dehn twists in $\mathrm{Mod}(\Sigma_1^k)$ for each $k \leq 9$. Here t_{δ_i} stands for the right-handed Dehn twist about a curve parallel to the i-th boundary component. We call this type of factorization a *k-holed torus relation* in general. Moreover, Tanaka [6] also gave such a factorization for $k = 8$, which turned out to be Hurwitz inequivalent to Korkmaz-Ozbagci's 8-holed torus relation. [1] Those factorizations have two major geometrical backgrounds, one as *Lefschetz pencils* and the other as *Stein fillings*, which demonstrate the fundamental importance of such k-holed torus relations.

It is a standard fact in the literature that the isomorphism classes of Lefschetz pencils are one-to-one correspondent to Dehn twist factorizations of the boundary multi-twists in the mapping class groups of holed surfaces, up to Hurwitz equivalence; a genus-g Lefschetz pencil with n critical points and k base points gives a monodromy factorization $t_{a_n} \cdots t_{a_1} = t_{\delta_1} \cdots t_{\delta_k}$ in the mapping class group of a k-holed surface of genus g, and conversely such a factorization determines a Lefschetz pencil. One can show that Korkmaz-Ozbagci's 9-holed torus relation yields a minimal elliptic Lefschetz pencil on $\mathbb{CP}^2$ (cf. [2]). For the other $k \leq 8$, Korkmaz-Ozbagci's k-holed torus relations simply correspond to Lefschetz pencils obtained by blowing-up some of the base points of the above pencil on $\mathbb{CP}^2$. On the other hand, it turns out that Tanaka's 8-holed torus relation yields a minimal elliptic Lefschetz pencil on $S^2 \times S^2$ (cf. [2]). Furthermore, it can be shown that a genus-1 Lefschetz pencil cannot have $k \geq 10$ base points (in fact, we might expect that the above pencils exhaust all possibilities). This is why Korkmaz and Ozbagci stopped at $k = 9$.

Dehn twist factorizations (with homologically nontrivial curves) of elements in mapping class groups of holed surfaces also provide positive allowable Lefschetz fibrations over D^2, which in turn represent Stein fillings of contact 3-manifolds. As summarized in [5], there is another elegant interpretation of the k-holed torus relations in this point of view. As the monodromy of an open book, the boundary multi-twist $t_{\delta_1} \cdots t_{\delta_k}$ in $\mathrm{Mod}(\Sigma_1^k)$ yields the contact 3-manifold (Y_k, ξ_k) that is given as the boundary of the symplectic D^2-bundle over T^2 with Euler number $-k$. While the symplectic D^2-bundle naturally gives a Stein filling of (Y_k, ξ_k), the positive allowable Lefschetz fibration over D^2 associated with the obvious Dehn

[1] The author also found 8-holed torus relations in [1, 2] in different contexts, which in fact can be shown to be Hurwitz equivalent to either Korkmaz-Ozbagci's or Tanaka's.

twist factorization $t_{\delta_1} \cdots t_{\delta_k}$ also gives the same Stein filling. If the boundary multi-twist $t_{\delta_1} \cdots t_{\delta_k}$ has another factorization (i.e. a k-holed torus relation) it also gives a Stein filling of (Y_k, ξ_k). Actually, Stein fillings of (Y_k, ξ_k) are already classified by Ohta and Ono [4]; besides the symplectic D^2-bundle there is (i) no more Stein filling when $k \geq 10$, (ii) one more Stein filling when $k \leq 9$ and $k \neq 8$, and (iii) two more Stein fillings when $k = 8$. Those Stein fillings can be realized by the positive allowable Lefschetz fibrations; the Stein filling in (ii) by Korkmaz-Ozbagci's k-holed torus relation, the two Stein fillings in (iii) by Korkmaz-Ozbagci's and Tanaka's 8-holed torus relations. The fact that (Y_k, ξ_k) has a unique Stein filling for $k \geq 10$ is the second reason that there is no $k(\geq 10)$-holed torus relation.

In addition to those theoretical importance, the k-holed torus relations are also practically useful in combinatorial constructions of new relations in the mapping class groups. In the first place, Tanaka [6] constructed his 8-holed torus relation in order to find new relations (which locate (-1)-sections of a well-known Lefschetz fibration) in higher genus mapping class groups. Similarly, the author [1] also constructed an 8-holed torus relation in the search of new relations (which give much small Lefschetz fibrations over tori).

Our aim in this note is to present simplified expressions for Korkmaz-Ozbagci's and Tanaka's k-holed torus relations. Although Korkmaz-Ozbagci gave satisfactorily simple expressions for $k \leq 4$, their curves become more involved as k increases. For the practical use of the k-holed torus relations, simpler expressions should be convenient for one who tries to use them.

In the rest of this paper, we basically follow the notation in [3] as follows. We will denote a right-handed Dehn twist along a curve α also by α. A left-handed Dehn twist along α will be denoted by $\bar{\alpha}$. We use the functional notation for multiplication; $\beta\alpha$ means we first apply α and then β. In addition, we denote the conjugation $\alpha\beta\bar{\alpha}$ by ${}_{\alpha}(\beta)$, which is the Dehn twist along the curve $t_\alpha(\beta)$.

1. Simplification

We will simplify Korkmaz-Ozbagci's 8- and 9-holed torus relations, and Tanaka's 8-holed torus relation. The other Korkmaz-Ozbagci's $k(\leq 7)$-holed torus relations can be obtained from the 9-holed torus relation by capping off some of the boundary components, therefore we do not give them explicitly.

With the curves shown in Figure 1(a), Korkmaz and Ozbagci [3] gave the 8-holed torus relation

$$\alpha_4\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7\beta_4\sigma_5 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8, \tag{1}$$

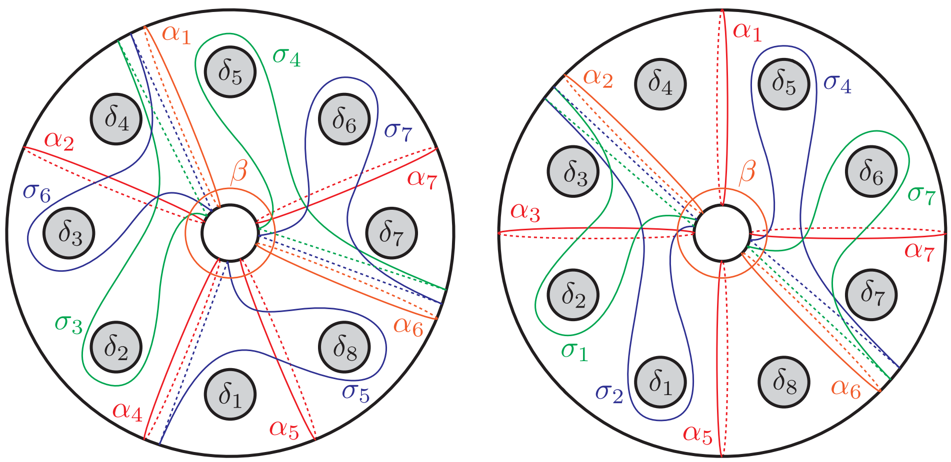
where $\beta_1 = {}_{\alpha_1}(\beta)$, $\beta_6 = {}_{\alpha_6}(\beta)$ and $\beta_4 = {}_{\alpha_4}(\beta)$. We modify this relation as follows:

$$\begin{aligned}
\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 &= \underline{\alpha_4\alpha_5}\ \underline{\beta_1\sigma_3\sigma_6}\alpha_2\underline{\beta_6\sigma_4\sigma_7}\alpha_7\underline{\beta_4\sigma_5} \\
&= \alpha_5\alpha_4{}_{\beta_1}(\sigma_3){}_{\beta_1}(\sigma_6)\underline{\beta_1\alpha_2}{}_{\beta_6}(\sigma_4){}_{\beta_6}(\sigma_7)\underline{\beta_6\alpha_7}{}_{\beta_4}(\sigma_5)\underline{\beta_4} \\
&= \underline{\beta_4\alpha_5}\alpha_4{}_{\beta_1}(\sigma_3){}_{\beta_1}(\sigma_6)\alpha_2{}_{\bar{\alpha}_2}(\beta_1){}_{\beta_6}(\sigma_4){}_{\beta_6}(\sigma_7)\alpha_7{}_{\bar{\alpha}_7}(\beta_6){}_{\beta_4}(\sigma_5) \\
&= \alpha_5{}_{\bar{\alpha}_5}(\beta_4)\alpha_4{}_{\beta_1}(\sigma_3){}_{\beta_1}(\sigma_6)\alpha_2{}_{\bar{\alpha}_2}(\beta_1){}_{\beta_6}(\sigma_4){}_{\beta_6}(\sigma_7)\alpha_7{}_{\bar{\alpha}_7}(\beta_6){}_{\beta_4}(\sigma_5).
\end{aligned}$$

By putting $a_1 = \alpha_5$, $b_1 = {}_{\bar{\alpha}_5}(\beta_4)$, $a_2 = \alpha_4$, $b_2 = {}_{\beta_1}(\sigma_3)$, $b_3 = {}_{\beta_1}(\sigma_6)$, $a_4 = \alpha_2$, $b_4 = {}_{\bar{\alpha}_2}(\beta_1)$, $b_5 = {}_{\beta_6}(\sigma_4)$, $b_6 = {}_{\beta_6}(\sigma_7)$, $a_7 = \alpha_7$, $b_7 = {}_{\bar{\alpha}_7}(\beta_6)$ and $b_8 = {}_{\beta_4}(\sigma_5)$, we have just obtained

$$a_1b_1a_2b_2b_3a_4b_4b_5b_6a_7b_7b_8 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8, \tag{2}$$

where the resulting curves are depicted in Figure 3(a). We refer to this relation as A_8.



(a) Korkmaz-Ozbagci's relation. (b) Tanaka's relation.

FIGURE 1. The curves for the two 8-holed torus relations.

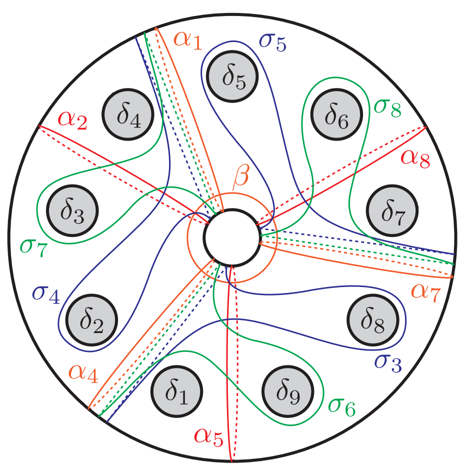


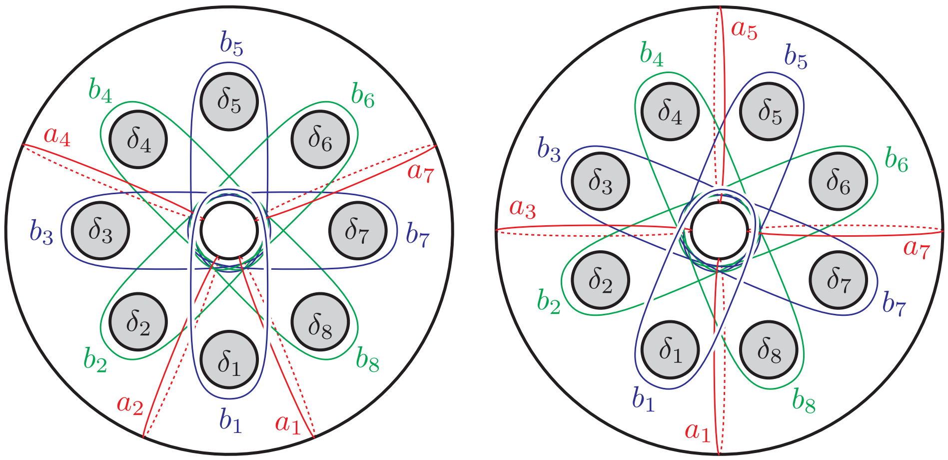
FIGURE 2. The curves for Korkmaz-Ozbagci's 9-holed torus relation.

With the curves shown in Figure 1(b), Tanaka [6] gave another 8-holed torus relation

$$\alpha_5\alpha_7\beta_{\bar{6}}\beta_2\sigma_2\sigma_1\alpha_1\alpha_3\beta_{\bar{2}}\beta_6\sigma_4\sigma_7 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8, \tag{3}$$

where $\beta_{\bar{6}} = {}_{\bar{\alpha}_6}(\beta)$, $\beta_2 = {}_{\alpha_2}(\beta)$, $\beta_{\bar{2}} = {}_{\bar{\alpha}_2}(\beta)$ and $\beta_6 = {}_{\alpha_6}(\beta)$. We modify this relation as follows:

$$\begin{aligned}
\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 &= \underline{\alpha_5\alpha_7}\beta_{\bar{6}}\beta_2\sigma_2\sigma_1\underline{\alpha_1\alpha_3}\beta_{\bar{2}}\beta_6\sigma_4\sigma_7\\
&= \alpha_7\underline{\alpha_5\beta_{\bar{6}}}\beta_2\sigma_2\sigma_1\alpha_3\underline{\alpha_1\beta_{\bar{2}}}\beta_6\sigma_4\sigma_7\\
&= \alpha_7\,{}_{\alpha_5}(\beta_{\bar{6}})\alpha_5\underline{\beta_2\sigma_2\sigma_1}\alpha_3\,{}_{\alpha_1}(\beta_{\bar{2}})\alpha_1\underline{\beta_6\sigma_4\sigma_7}\\
&= \alpha_7\,{}_{\alpha_5}(\beta_{\bar{6}})\alpha_5\,{}_{\beta_2}(\sigma_2)\,{}_{\beta_2}(\sigma_1)\beta_2\alpha_3\,{}_{\alpha_1}(\beta_{\bar{2}})\alpha_1\,{}_{\beta_6}(\sigma_4)\,{}_{\beta_6}(\sigma_7)\beta_6\\
&= \alpha_5\,{}_{\beta_2}(\sigma_2)\,{}_{\beta_2}(\sigma_1)\underline{\beta_2\alpha_3}\,{}_{\alpha_1}(\beta_{\bar{2}})\alpha_1\,{}_{\beta_6}(\sigma_4)\,{}_{\beta_6}(\sigma_7)\underline{\beta_6\alpha_7}\,{}_{\alpha_5}(\beta_{\bar{6}})\\
&= \alpha_5\,{}_{\beta_2}(\sigma_2)\,{}_{\beta_2}(\sigma_1)\alpha_3\,{}_{\bar{\alpha}_3}(\beta_2)\,{}_{\alpha_1}(\beta_{\bar{2}})\alpha_1\,{}_{\beta_6}(\sigma_4)\,{}_{\beta_6}(\sigma_7)\alpha_7\,{}_{\bar{\alpha}_7}(\beta_6)\,{}_{\alpha_5}(\beta_{\bar{6}}).
\end{aligned}$$



(a) The relation A_8. (b) The relation B_8.

FIGURE 3. The curves for the simplified 8-holed torus relations.

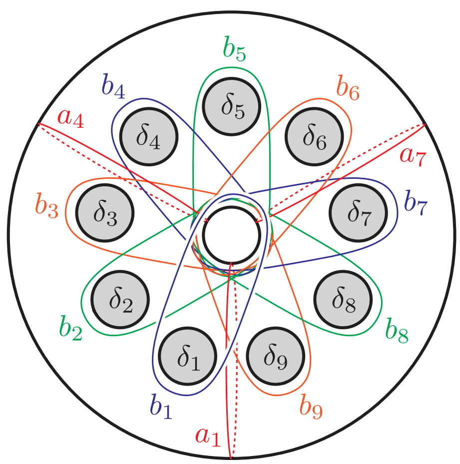


FIGURE 4. The curves for the simplified 9-holed torus relation A_9.

By putting $a_1 = \alpha_5$, $b_1 = {}_{\beta_2}(\sigma_2)$, $b_2 = {}_{\beta_2}(\sigma_1)$, $a_3 = \alpha_3$, $b_3 = {}_{\bar{\alpha}_3}(\beta_2)$, $b_4 = {}_{\alpha_1}(\beta_{\bar{2}})$, $a_5 = \alpha_1$, $b_5 = {}_{\beta_6}(\sigma_4)$, $b_6 = {}_{\beta_6}(\sigma_7)$, $a_7 = \alpha_7$, $b_7 = {}_{\bar{\alpha}_7}(\beta_6)$ and $b_8 = {}_{\alpha_5}(\beta_{\bar{6}})$, we obtained

$$a_1 b_1 b_2 a_3 b_3 b_4 a_5 b_5 b_6 a_7 b_7 b_8 = \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8, \tag{4}$$

where the resulting curves are depicted in Figure 3(b). We refer to this relation as B_8.

Remark 1. Let $X_{A_8} \to D^2$ and $X_{B_8} \to D^2$ be the positive allowable Lefschetz fibrations associated with A_8 and B_8, respectively. As observed in [5], we have $H_1(X_{A_8}; \mathbb{Z}) = 0$ and $H_1(X_{B_8}; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, hence X_{A_8} and X_{B_8} give the two distinct Stein fillings (other than the D^2-bundle) of the contact 3-manifold (Y_8, ξ_8).

With the curves shown in Figure 2, Korkmaz and Ozbagci [3] gave the 9-holed torus relation

$$\beta_4\sigma_3\sigma_6\alpha_5\beta_1\sigma_4\sigma_7\alpha_2\beta_7\sigma_5\sigma_8\alpha_8 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9, \tag{5}$$

where $\beta_4 = {}_{\alpha_4}(\beta)$, $\beta_1 = {}_{\alpha_1}(\beta)$ and $\beta_7 = {}_{\alpha_7}(\beta)$. We modify this relation as follows:

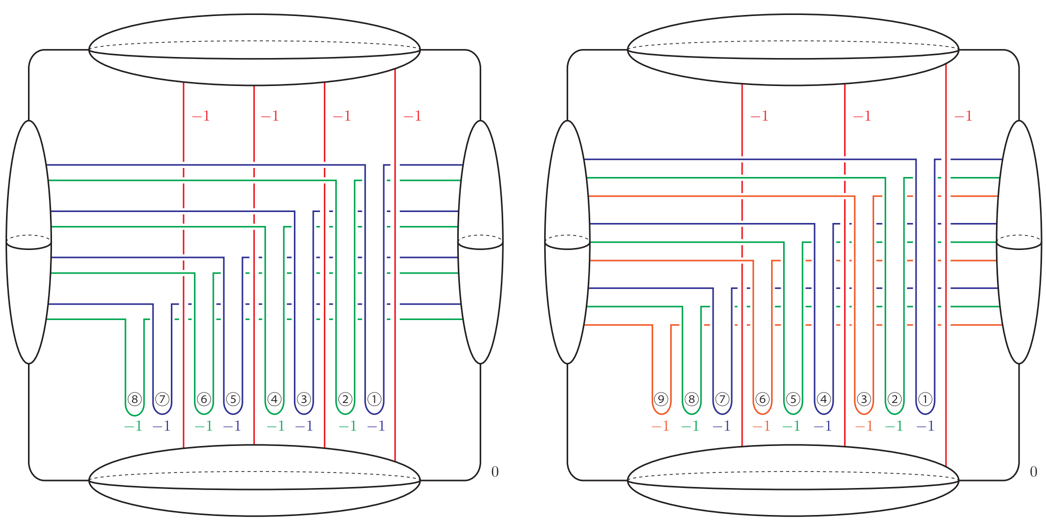
$$\begin{aligned}
\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9 &= \underline{\beta_4\sigma_3\sigma_6}\alpha_5\underline{\beta_1\sigma_4\sigma_7}\alpha_2\underline{\beta_7\sigma_5\sigma_8}\alpha_8 \\
&= {}_{\beta_4}(\sigma_3){}_{\beta_4}(\sigma_6)\underline{\beta_4\alpha_5}{}_{\beta_1}(\sigma_4){}_{\beta_1}(\sigma_7)\underline{\beta_1\alpha_2}{}_{\beta_7}(\sigma_5){}_{\beta_7}(\sigma_8)\underline{\beta_7\alpha_8} \\
&= {}_{\beta_4}(\sigma_3){}_{\beta_4}(\sigma_6)\alpha_5{}_{\bar{\alpha}_5}(\beta_4){}_{\beta_1}(\sigma_4){}_{\beta_1}(\sigma_7)\alpha_2{}_{\bar{\alpha}_2}(\beta_1){}_{\beta_7}(\sigma_5){}_{\beta_7}(\sigma_8)\alpha_8{}_{\bar{\alpha}_8}(\beta_7) \\
&= \alpha_5{}_{\bar{\alpha}_5}(\beta_4){}_{\beta_1}(\sigma_4){}_{\beta_1}(\sigma_7)\alpha_2{}_{\bar{\alpha}_2}(\beta_1){}_{\beta_7}(\sigma_5){}_{\beta_7}(\sigma_8)\alpha_8{}_{\bar{\alpha}_8}(\beta_7){}_{\beta_4}(\sigma_3){}_{\beta_4}(\sigma_6).
\end{aligned}$$

By putting $a_1 = \alpha_5$, $b_1 = {}_{\bar{\alpha}_5}(\beta_4)$, $b_2 = {}_{\beta_1}(\sigma_4)$, $b_3 = {}_{\beta_1}(\sigma_7)$, $a_4 = \alpha_2$, $b_4 = {}_{\bar{\alpha}_2}(\beta_1)$, $b_5 = {}_{\beta_7}(\sigma_5)$, $b_6 = {}_{\beta_7}(\sigma_8)$, $a_7 = \alpha_8$, $b_7 = {}_{\bar{\alpha}_8}(\beta_7)$, $b_8 = {}_{\beta_4}(\sigma_3)$ and $b_9 = {}_{\beta_4}(\sigma_6)$, we obtained

$$a_1b_1b_2b_3a_4b_4b_5b_6a_7b_7b_8b_9 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9, \tag{6}$$

where the resulting curves are depicted in Figure 4. We refer to this relation as A_9.

Remark 2. As we mentioned earlier, the relations B_8 and A_9 correspond to minimal Lefschetz pencils on $S^2 \times S^2$ and $\mathbb{CP}^2$, respectively (while the others, including A_8, are just blow-up of the latter). In Figure 5, we draw two handle decompositions of the elliptic Lefschetz fibration $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2 \to S^2$ and locate the (-1)-sections corresponding to B_8 and A_9. Blowing down those sections must yield the 4-manifolds $S^2 \times S^2$ and $\mathbb{CP}^2$, respectively, and the exceptional spheres become the base points of the Lefschetz pencils.



(a) The eight (-1)-sections corresponding to B_8. (b) The nine (-1)-sections corresponding to A_9.

FIGURE 5. Handle decompositions of the elliptic Lefschetz fibration $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2 \to S^2$ with configurations of (-1)-sections.

Acknowledgements. The author thanks to Kenta Hayano and Takahiro Oba for informing him of the aspect of Stein fillings about k-holed torus relations.

References

[1] N. Hamada, *Upper bounds for the minimal number of singular fibers in a Lefschetz fibration over the torus*, Michigan Math. J. **63** (2014), 275-291.

[2] N. Hamada, *Sections of the Matsumoto-Cadavid-Korkmaz Lefschetz fibration*, preprint; https://arxiv.org/abs/1610.08458

[3] M. Korkmaz and B. Ozbagci, *On sections of elliptic fibrations*, Michigan Math. J. **56** (2008), 77–87.

[4] H. Ohta and K. Ono, *Symplectic fillings of the link of simple elliptic singularities*, J. Reine Angew. Math. **565** (2003), 183-205.

[5] B. Ozbagci, *On the topology of fillings of contact* 3*-manifolds*, Geometry & Topology Monographs Vol. **19** (2015), 73–123.

[6] S. Tanaka, *On sections of hyperelliptic Lefschetz fibrations*, Algebr. Geom. Topol. **12** (2012), 2259–2286.

Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Meguro-ku, Tokyo 153-8914, Japan

E-mail address: nhamada@ms.u-tokyo.ac.jp